A study of equilibrium states of homogeneous turbulence submitted to an inclined shear

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Abstract

The aim of this work is to compare the prediction of several second order models in the results of direct numerical simulations of Jacobitz [1]. The second order models retained for this work are the classic models of Launder, Reece and Rodi (LRR) [2], Craft and Launder (CL) [3,4], Shih and Lumley (SL) [5] and the Speziale, Sarkar and Gatski (SSG) model [6]. The main objective is the prediction of the equilibrium states of dimensionless parameters characterizing the homogeneous flows considered and for a turbulence submitted to an inclined shear ($\theta = \pi / 4$). A non dimensional form of the evolution equations have been obtained after modelling and introducing kinematics and scalars dimensionless parameters. A fourth order Runge–Kutta method is used for the integration of the modeled differential equations sumitted to the initial conditions of the results of direct numerical simulations of Jacobitz. A general tendency to asymptotic equilibrium states for the dimensionless parameters has been observed.

Keywords: Second order models, Asymptotic equilibrium states, Stratified turbulence, Inclined shear.

Nomenclature

b	Anisotropic tensor of Reynolds $b_{ij} = \overline{u_i u_j} / 2k - \delta_{ij} / 3$
U_i	I-th component of mean velocity (m.s^{-1})
u_i	I-th component of the fluctuating velocity (m.s^{-1})
P	Pressure (N.m^{-2})
p	Fluctuation of the pressure (N.m^{-2})
S	Mean shear (s^{-1})
S_ρ	Mean scalar gradient ($^\circ$C.m^{-1})
g	Constant of gravity
t	Time (s)
R_i	Dimensionless Richardson number $R_i = \frac{g}{T_0} \frac{S_\rho}{S^2}$
C_p	Chaleur massique of constant pressure (J K^{-1} mol^{-1})
$\overline{u_i u_j}$	Reynolds stress tensor (m^{2}s^{-2})
$\overline{u_i \rho}$	Turbulent flux of the scalar (m$^\circ$C s^{-1})
K	Turbulent kinetic energy k=$\overline{u_i u_i} / 2$ (m^{2}s^{-2})
$\overline{U}_{p,q}$	Gradient of mean speed (s^{-1})
$\overline{T}_{,i}$	Gradient of the scalar ($^\circ$C.m$^{-1)}$
x_i	Component of an orthonormal catesian coordinate system (m)

Greek letters :

α	Thermal diffusivity
μ	Viscosity of flux (kg m^{-1} s^{-1})
ν	Kinematic viscosity $\nu = \dfrac{\mu}{\rho}$ (m^2 s^{-1})
λ	Conductibilité of the scalar (W m^{-1} $^{\circ}$C^{-1})
β	Terms of gravité $\beta = \dfrac{g}{T_0}$ (m^2 s^{-2} $^{\circ}$C^{-1})
τ	Non dimensional time $\tau = St$
δ_{ij}	Kronecker Symbol
ρ	Fluctuation of the scalar ($^{\circ}$C)
ρ_0	Density of reference (kg m^{-3})
$\overline{\rho^2}$	Scalar of variance ($^{\circ}$C^2)
ε	Terms of dissipation of turbulent kinetic energy (m^2 s^{-3})
$\varepsilon_{\rho\rho}$	Terms of dissipation of variance of the scalar ($^{\circ}$C^2 s^{-1})
ε_{ij}	Terms of dissipation of tensor of Reynolds (m2 s^{-3})
$\varepsilon_{i\rho}$	Terms of dissipation of the scalar flux turbulent (m $^{\circ}$C s^{-2})
ϕ_{ij}	Terms of pressure-strain correlation (m^2 s^{-3})
$\phi_{i\rho}$	Terms of pressure- scalar gradient correlation ($^{\circ}$C m s^{-2})

Subscripts :

SSG	Speziale,Sarkar and Gatski model
SL	Shih and Lumley model
CL	Craft and Launder model
LRR	Launder, Reece and Rodi model
DNS	Direct Numerical Simulation
RK4	Fourth order Runge-Kutta method

1. Introduction

The study and the prediction of the equilibrium states of homogeneous turbulence has been the subject of a variety of experimental, computational, and theoretical studies during the past four decades. The popularity of this flow lies in the fact that it accounts for an important physical effect the alteration of the turbulence structure by shear in a simplified setting unencumbered by such complications as rigid boundaries and mean turbulent diffusion. With this framework, the study carried out by Speziale et al. [7] constitute a reference essential for the prediction of the eqilibrium values in sheared homogeneous turbulence. Von Karman [8]
first proposed the problem of homogeneous shear flow which gave rise to some mathematical studies. The first successful experimental realization of homogeneous shear flow in the laboratory was achieved by Rose [9] and was then followed by a series of landmark experiments by Champagne et al. [10] and Harris et al. [11]. In fact, equilibrium states of several benchmark homogeneous shear flows have been used in the development and analysis of turbulence models [12,13].
In recent years, research has been focused on the study of homogenous stably stratified shear flows. Numerous laboratory experiments on vertically stably stratified homogenous shear flows have been performed. On the other hand, numerical simulations of such flows have been carried out by Gerz et al. [14] and Holt et al. [15]. The global parameter that measures the relative influence of shear and stratification on the flow is the gradient Richardson number $R_i = \dfrac{N^2}{S^2}$, where $N^2 = -\dfrac{g\,S_\rho}{\rho_0}$ is the Brunt-Väisälä frequency and $S = \sqrt{\left(d\overline{U}_1 / dx_2\right)^2 + \left(d\overline{U}_1 / dx_3\right)^2}$ is the shear rates. An important property of homogeneous turbulent flows is the appearance of dynamic state variables that tend to approach equilibrium values in the long time limit. The equilibrium states provide an important benchmark in the calibration of closure models. For homogeneous turbulent flows without buoyancy effects, the fixed points associated with the equilibrium can be

determined [16], these fixed points can then be used to assess the suitability of higher-order models and their ability to predict the correct equilibrium values. For example, Abid et al. [17] calculated the equilibrium states for homogeneous turbulent shear flow and channel flow using Reynolds stress closures in order to assess their predictive performance. Tavoularis et al. [18] conjectured that equilibrium states also exist for buoyant shear flows; however they take a longer time to achieve due to the interaction between shear and buoyancy. The approach to equilibrium turbulence for buoyant shear flow is much more complicated. Even the parameters that characterize the equilibrium states are to date not known precisely.

During last years, Gerz et al. [14] were interested in the study of the direct digital simulations [19-20] for a homogeneous turbulence submitted to a vertical shear. They showed the influence of the Richardson number Ri on the turbulent sizes. Komori et al. [21] studied the case of a laminated flow in an opened water channel. Itsweire et al. [22] showed the importance of the Richardson number Ri on the evolution of turbulence. However, a more significant interest was allotted in the study of the equilibrium states of a turbulence presenting a stable stratification and a non-vertical shear [23]. In this context, only Jacobitz and al. [20] analysed the influence of the angle of shear inclination θ compared to the vertical on the evolution of the various turbulent parameters. They showed the importance of the equilibrium states associated with the non-dimensional kinematics and scalars parameters.

The aims of this study is to confirm the existence of an asymptotic equilibrium states of a turbulence submitted to an inclined shear (θ=π/4). For this, the solutions obtained parameterized by Richardson number Ri, confirm the asymptotic behaviors of the non-dimensional kinematics and scalars parameters. Two of the most known second order models are retained for the pressure strain correlation, pressure scalar gradient correlation and time evolution equation of kinematic and thermal dissipations. The Speziale Sarkar and Gatski (SSG) model is retained for the terms of the pressure-strain correlation and is coupled with some classical turbulence models for the pressure scalar correlation, namely the Craft and Launder (CL) model, the Shih and Lumley (SL) model and the Launder, Reece and Rodi (LRR) model. A pecular attention is accorded to the contribution of the SSG model on the modeling of turbulent shear flow. Our result will be compared to previous result of litterature, such as direct numirical simulation (DNS) of Jacobitz [24] and Jacobitz et al. [20].

2. Theoretical Background

2.1 The governing equations

The density ρ, the velocity u_i, and the pressure p, denote fluctuations with respect to the mean density $\overline{\rho}$, the mean velocity $\overline{U}$, and the mean pressure $\overline{p}$. The uniform mean density gradient $\frac{\partial \overline{\rho}}{\partial x_3} = S_\rho$ imposes a stable stratification which is hydrostatically balanced by a corresponding mean pressure gradient. The mean velocity $\overline{U} = (\overline{U}_1, 0, 0)$ has two constant shear rates, a horizontal $\frac{\partial \overline{U}_1}{\partial x_2} = S_2 = S \sin\theta$ and a vertical one $\frac{\partial \overline{U}_1}{\partial x_3} = S_3 = S \cos\theta$. Here θ designates the angle between the shear and the vertical gradient of stratification $\frac{\partial \overline{\rho}}{\partial x_3}$. Thus, $\overline{U}_i = (S \sin\theta\, x_2 + S\cos\theta\, x_3)\delta_{i1}$. After the customary Boussinesq assumption, the equations governing the evolution of the fluctuating variables are as follows:

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + (S\sin\theta\, x_2 + S\cos\theta\, x_3)\frac{\partial u_i}{\partial x_1} + (S\sin\theta\, u_2 + S\cos\theta\, u_3)\delta_{i1} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - \frac{g}{\rho_0}\rho\delta_{i3} \tag{2}$$

$$\frac{\partial \rho}{\partial t} + u_k \frac{\partial \rho}{\partial x_k} + (S\sin\theta\, x_2 + S\cos\theta\, x_3)\frac{\partial \rho}{\partial x_1} + S_\rho u_3 = \alpha \frac{\partial^2 \rho}{\partial x_k \partial x_k} \tag{3}$$

2.2 Second-order equations

The exact Reynolds stress transport equation is given by:

$$\frac{d\,\overline{u_i\,u_j}}{dt} = P_{ij} - B_{ij} + \phi_{ij} - \varepsilon_{ij} \tag{4}$$

The terms, in the above equation, respectively, are the time rate of change, production due to mean shear (P_{ij}), term of gravity (B_{ij}), pressure strain correlation (ϕ_{ij}) and viscous dissipation rate (ε_{ij}) of Reynolds stress:

$$P_{ij} = -S\sin\theta\,\overline{u_j\,u_2}\,\delta_{i1} - S\cos\theta\,\overline{u_j\,u_3}\,\delta_{i1} - S\sin\theta\,\overline{u_i\,u_2}\,\delta_{j1} - S\cos\theta\,\overline{u_i\,u_3}\,\delta_{j1} \tag{5}$$

$$B_{ij} = \frac{g}{\rho_0}\left(\overline{u_i\,\rho}\,\delta_{j3} + \overline{u_j\,\rho}\,\delta_{i3}\right) \tag{6}$$

$$\phi_{ij} = \frac{1}{\rho_0}\,\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \tag{7}$$

$$\varepsilon_{ij} = 2\nu\,\overline{\frac{\partial u_i}{\partial x_k}\,\frac{\partial u_j}{\partial x_k}} \tag{8}$$

Given that ν is the kinematic viscosity, ρ_0 is the reference density of the fluid and β is the thermal expansion coefficient. Hence, the production terms are exact but the terms ϕ_{ij} and ε_{ij} require modeling.

Transport equations for the the components $\overline{u_i\,\rho}$ of the turbulent scalar flux is:

$$\frac{d\overline{u_i\,\rho}}{dt} = P_{i\rho} - B_{i\rho} + \phi_{i\rho} - \varepsilon_{i\rho} \tag{9}$$

The terms, in the above equation, correspond to the time rate of change, thermal production ($P_{i\rho}$), term of gravity ($B_{i\rho}$), pressure–temperature gradient correlation ($\phi_{i\rho}$) and viscous dissipation ($\varepsilon_{i\rho}$):

$$P_{i\rho} = -S\sin\theta\,\overline{u_2\,\rho}\,\delta_{i1} - S\cos\theta\,\overline{u_3\,\rho}\,\delta_{i1} - S_\rho\,\overline{u_i\,u_3} \tag{10}$$

$$B_{i\rho} = \frac{g}{\rho_0}\,\overline{\rho^2}\,\delta_{i3} \tag{11}$$

$$\phi_{i\rho} = \frac{1}{\rho_0}\,\overline{p\,\frac{\partial\rho}{\partial x_i}} \tag{12}$$

$$\varepsilon_{i\rho} = (\alpha+\nu)\,\overline{\frac{\partial\rho}{\partial x_k}\,\frac{\partial u_i}{\partial x_k}} \tag{13}$$

A transport equations for the density variance $\overline{\rho^2}$ is derived from the transport equation of density (3) :

$$\frac{d\,\overline{\rho^2}}{dt} = P_{\rho\rho} - 2\varepsilon_{\rho\rho} \tag{14}$$

$$P_{\rho\rho} = -2S_\rho\,\overline{\rho\,u_3} \tag{15}$$

$$\varepsilon_{\rho\rho} = -2\nu\,\overline{\frac{\partial\rho}{\partial x_k}\,\frac{\partial\rho}{\partial x_k}} \tag{16}$$

In the above equation, the term $P_{\rho\rho}$ is the production of temperature variance and the term $\varepsilon_{\rho\rho}$ is the viscous dissipation rate of temperature variance.

While considering the trace of the equation (4), we get the equation of evolution of the turbulent kinetic energy:

$$\frac{dK}{dt} = P_2 + P_3 - B - \varepsilon \tag{17}$$

Where P_2 is the turbulence production term due to horizontal shear $\frac{\partial \overline{U_1}}{\partial x_2}$.

$$P_2 = -S \sin\theta \overline{u_1 u_2} \quad (18)$$

P_3 is the turbulence production term due to vertical shear $\frac{\partial \overline{U_1}}{\partial x_3}$.

$$P_3 = -S \cos\theta \overline{u_1 u_3} \quad (19)$$

B the buoyancy term

$$B = \frac{g}{\rho_0} \overline{u_3 \rho} \quad (20)$$

and ε is the dissipation term.

$$\varepsilon = \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial ui}{\partial x_k}} \quad (21)$$

From the density variance, the potential energy K_ρ can be computed:

$$K_\rho = \frac{1}{2} \frac{g}{\rho_0 S_\rho} \overline{\rho^2} \quad (22)$$

In cartesian notation, equations (4), (9), (14) and (17) are written on the following form:

$$\frac{d\overline{u^2{}_1}}{dt} = -2S \sin\theta \overline{u_1 u_2} - 2S \cos\theta \overline{u_1 u_3} + \frac{1}{\rho_0} \overline{p\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{x_1}\right)} - 2\gamma \overline{\frac{\partial u_1}{\partial x_k} \frac{\partial u_1}{\partial x_k}} \quad (23)$$

$$\frac{d\overline{u^2{}_2}}{dt} = \frac{1}{\rho_0} \overline{p\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{x_2}\right)} - 2\gamma \overline{\frac{\partial u_2}{\partial x_k} \frac{\partial u_2}{\partial x_k}} \quad (24)$$

$$\frac{d\overline{u^2{}_3}}{dt} = -2\frac{g}{\rho_0} \overline{u_3 \rho} + \frac{1}{\rho_0} \overline{p\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{x_3}\right)} - 2\gamma \overline{\frac{\partial u_3}{\partial x_k} \frac{\partial u_3}{\partial x_k}} \quad (25)$$

$$\frac{d\overline{u_1 u_2}}{dt} = -S \sin\theta \overline{u_2}^2 - S \cos\theta \overline{u_2 u_3} + \frac{1}{\rho_0} \overline{p\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{x_1}\right)} \quad (26)$$

$$\frac{d\overline{u_1 u_3}}{dt} = -S \sin\theta \overline{u_2 u_3} - S \cos\theta \overline{u_3}^2 - \frac{g}{\rho_0} \overline{u_1 \rho} + \frac{1}{\rho_0} \overline{p\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{x_1}\right)} \quad (27)$$

$$\frac{d\overline{u_2 u_3}}{dt} = -\frac{g}{\rho_0} \overline{u_2 \rho} + \frac{1}{\rho_0} \overline{p\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{x_2}\right)} \quad (28)$$

$$\frac{d\overline{u_1 \rho}}{dt} = -S \sin\theta \overline{u_2 \rho} - S \cos\theta \overline{u_3 \rho} - S_\rho \overline{u_1 u_3} + \frac{1}{\rho_0} \overline{p \frac{\partial \rho}{\partial x_1}} \quad (29)$$

$$\frac{d\overline{u_2 \rho}}{dt} = -S_\rho \overline{u_2 u_3} + \frac{1}{\rho_0} \overline{p \frac{\partial \rho}{\partial x_2}} \quad (30)$$

$$\frac{d\overline{u_3 \rho}}{dt} = -S_\rho \overline{u_3^2} - \frac{g}{\rho_0} \overline{\rho^2} + \frac{1}{\rho_0} \overline{p \frac{\partial \rho}{\partial x_3}} \quad (31)$$

$$\frac{d\overline{\rho^2}}{dt} = -2S_\rho \overline{u_3 \rho} - 2\alpha \overline{\frac{\partial \rho}{\partial x_k} \frac{\partial \rho}{\partial x_k}} \quad (32)$$

$$\frac{dK}{dt} = -S\sin\theta\,\overline{u_1 u_2} - S\cos\theta\,\overline{u_1 u_3} - \frac{g}{\rho_0}\overline{u_3\,\rho} - \nu\,\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} \quad (33)$$

3. Second-order modelling

Second-order turbulence closure models are of great utility for the prediction of geophysical and engineering flows. The pressure–strain correlation ϕ_{ij} , and the pressure–scalar gradient correlation $\phi_{i\rho}$ are besides the time evolutions of the dissipations ε and ε_{ρ} , the principal terms to be modelled in a second-order closure modelling of the flow considered here.

In a stratified shear flow, the correlations ϕ_{ij} and $\phi_{i\rho}$ are classically decomposed into three contributions [25]:

$$\phi_{ij} = \phi_{ij}^1 + \phi_{ij}^2 + \phi_{ij}^3 \quad (34)$$

$$\phi_{i\rho} = \phi_{i\rho}^1 + \phi_{i\rho}^2 + \phi_{i\rho}^3 \quad (35)$$

Where the contributions 1 are the turbulent-turbulent terms, the terms 2 represent the interaction between mean and turbulent flows. The terms 3 are terms due to buoyancy.

3.1 The pressure strain-correlation

Sarkar et al. [26] developed a model of the pressure strain correlation which take into account the condition of realisability [27] of Schumann and Shih and Lumley [28] and predict stable fixed points.

$$\phi_{ij}^{\ 1} = -C_1\, b_{ij} + 3(C_1 - 2)(b_{ij}^2 + II_b\,\frac{\delta_{ij}}{3}), \quad II_b = b_{kl}b_{lk} \quad (36)$$

$$\phi_{ij}^{\ 2} = C_2\, b_{mn}\, S_{mn}\, b_{ij} + \frac{1}{2}(C_3 - \sqrt{b_{mn}\, b_{mn}}\, C_{33})\, S_{ij} + \frac{1}{2}C_4(b_{ik}S_{kj} + b_{jk}S_{ki} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}) + \frac{1}{2}C_5(b_{ik}w_{jk} + b_{jk}w_{ik}) \quad (37)$$

Where numerical values of constants are:

$C_1 = 3.4 \quad C_2 = 1.8 \quad C_3 = 0.8 \quad C_4 = 1.25 \quad C_5 = 0.4 \quad C_{33} = 1.3$

The Zeman and Lumley model [29] has been retained for the buoyancy terms in pressure strain correlation :

$$\phi_{ij}^3 = -C_3\left(\beta_j\,\overline{u_i\,\rho} + \beta_i\,\overline{u_j\,\rho} - \frac{2}{3}\beta_l\,\overline{u_l\,\rho}\,\delta_{ij}\right),\ C_3 = 0.5 \quad (38)$$

$$\phi_{i\rho}^3 = -C_{3\rho}\,\beta_i\,\overline{\rho^2}\ ,\ C_{3\rho} = 0.5 \quad (39)$$

3.2 The pressure scalar gradient correlation

The SSG second order model for ϕ_{ij} retained for the kinematic field has no extension to the scalar field. To obtain a system of closed equations, the Speziale, Sarkar and Gatski [6] model for the kinematic field is coupled with the Launder, Reece and Rodi model [3], the Shih and Lumley model (SL) [30] model and the Craft and Launder (CL) [31] model for the pressure scalar gradient correlation respectively.

3.2.1 The Speziale, Sarkar and Gatski (SSG) model

The Speziale, Sarkar et Gatski (SSG) second order closure model has know a great success during the last two decades. It has been submitted in addition to the kinematics constraints to the strong form of realisability. Furthermore it predicts stable fixed points [32]. The final form of this model is the followed one:

$$\phi_{ij} = \phi_{ij}^1 + \phi_{ij}^2$$

$$\phi_{ij}^{\ 1} = -C_1\, b_{ij} + 3(C_1 - 2)(b_{ij}^2 + II_b\,\frac{\delta_{ij}}{3}), \quad II_b = b_{kl}b_{lk}\ ,\ C_1 = 3.4 \quad (40)$$

$$\phi_{ij}{}^{2} = C_2\, b_{mn}\, S_{mn}\, b_{ij} + \frac{1}{2}(C_3 - \sqrt{b_{mn}\, b_{mn}}\, C_{33})\, S_{ij} + \frac{1}{2} C_4 (b_{ik} S_{kj} + b_{jk} S_{ki} - \frac{2}{3} b_{mn} S_{mn} \delta_{ij}) + \frac{1}{2} C_5 (b_{ik} w_{jk} + b_{jk} w_{ik}) \tag{41}$$

Where $C_2 = 1.8 \quad C_3 = 0.8 \quad C_4 = 1.25 \quad C_5 = 0.4 \quad C_{33} = 1.3$

3.2.2 The Craft and Launder (CL) model

Craft and launder (CL) [4] developed a second-order closure model of pressure-strain and pressure-scalar gradient correlation [4, 31] retained the condition of two-dimensional turbulence which is summarized with the component normal speed to the wall is cancelled more quickly than the other components of the fluctuation speed. It is translated for the heat flux $\overline{u_3\,\rho}$ by :

$\frac{d\,\overline{u_3\,\rho}}{dt} = 0$ When $u_3 \to 0$

$$\phi_{i\rho}^{1} = -3{,}2 \frac{\varepsilon}{k} r \left(1 + \mathrm{II}^{0.5}\right)\left(\overline{u_i\,\rho} - 2{,}2 b_{ik}\, \overline{u_k\,\rho} + 6{,}4 b_{ik} b_{kj}\, \overline{u_j\,\rho}\right) \tag{42}$$

Where $r = \frac{\varepsilon_\rho}{\varepsilon} \frac{k}{\overline{\rho^2}}$

$$\begin{aligned}\phi_{i\rho}^{2} = {} & 0{,}8\overline{\rho u_k}\frac{\partial U_i}{\partial x_k} - 0{,}2\overline{\rho u_k}\frac{\partial U_k}{\partial x_i} + \frac{\varepsilon}{k}\overline{\rho u_i}\left(\frac{P_{kk}}{\varepsilon}\right) - 0{,}4\overline{\rho u_k}\, a_{il}\left[\frac{\partial U_k}{\partial x_l} + \frac{\partial U_l}{\partial x_k}\right] \\ & + 0{,}1\overline{\rho u_k}\, a_{ik}\, a_{ml}\left[\frac{\partial U_m}{\partial x_l} + \frac{\partial U_l}{\partial x_m}\right] - 0{,}1\overline{\rho u_k}\left(a_{im}\, P_{mk} + 2 a_{mk}\, P_{im}\right) \\ & - 0{,}05 a_{ml}\left\{7 a_{mk}\left[\overline{\rho u_i}\frac{\partial U_k}{\partial x_l} + \overline{\rho u_k}\frac{\partial U_i}{\partial x_l}\right] - \overline{\rho u_k}\left[a_{ml}\frac{\partial U_i}{\partial x_k} + a_{mk}\frac{\partial U_i}{\partial x_l}\right]\right\} \\ & + 0{,}15 a_{mk}\left[\frac{\partial U_k}{\partial x_l} + \frac{\partial U_l}{\partial x_k}\right]\left(a_{mk}\overline{\rho u_i} - a_{mi}\overline{\rho u_k}\right)\end{aligned} \tag{43}$$

3.2.3 The Shih and Lumley (SL) model

The application of the strict condition of realisability by Shih and Lumley model separately to the first and second terms of the pressure-strain and pressure-scalar gradient correlations ϕ_{ij} *and* $\phi_{i\rho}$ respectively leads to abordable and interesting froms of these models.

The final form of the model of correlation is $\phi_{i\rho} = \phi_{i\rho}^{1} + \phi_{i\rho}^{2}$

$$\phi_{i\rho}^{1} = -C_{1\rho}\frac{\varepsilon}{k}\left(\overline{u_i\,\rho} + C_{1\rho}^{'}\, a_{ik}\, \overline{u_k\,\rho}\right), a_{ik} = 2 b_{ik} \tag{44}$$

$$\phi_{i\rho}^{2} = B_{jk}^{i}\, U_{j,k} \tag{45}$$

$$\begin{aligned}B_{jk}^{i} = {} & \alpha_1 \overline{\rho u_i}\,\delta_{jk} + \alpha_2\left(\delta_{ik}\,\overline{\rho u_j} + \delta_{jk}\,\overline{\rho u_i}\right) + \alpha_3 b_{ik}\overline{\rho u_j} + \alpha_4\left(b_{ij}\overline{\rho u_k} + b_{jk}\overline{\rho u_i}\right) \\ & + \alpha_5\left(\delta_{ij} b_{kp} + \delta_{kj} b_{kp}\right)\overline{\rho u_p} + \alpha_6 \delta_{ik} b_{jp}\overline{\rho u_p} + \alpha_7 b_{ik} b_{jp}\overline{\rho u_i} + \alpha_8\left(b_{ij} b_{kp} + b_{kj} b_{ip}\right)\overline{\rho u_p} \\ & + \alpha_9 b_{ik}^2\overline{\rho u_j} + \alpha_{10}\left(b_{ij}^2\overline{\rho u_k} + b_{jk}^2\overline{\rho u_i}\right) + \alpha_{11}\delta_{ik} b_{jp}^2\overline{\rho u_p} + \alpha_{12}\left(\delta_{ij} b_{kp}^2 + \delta_{ij} b_{ij}^2\right)\overline{\rho u_p}\end{aligned}$$

With

$\alpha_1 = 0{,}8$	$\alpha_5 = 0$	$\alpha_9 = 0$
$\alpha_2 = -0{,}2$	$\alpha_6 = 0{,}2$	$\alpha_{10} = 0$
$\alpha_3 = 0{,}1$	$\alpha_7 = 0$	$\alpha_{11} = 0$
$\alpha_4 = -0{,}3$	$\alpha_8 = 0$	$\alpha_{12} = 0$

3.2.4 The Launder, Reece and Rodi (LRR) model

The LRR model [2] is most known for the pressure-strain correlation.

The $\phi^1_{i\rho}$ term is the contribution non linear, as of return to the isotropy, which is written according to the turbulent $\overline{\rho u_i}$ heat flux in the following form:

$$\phi^1_{i\rho} = -C_1^{'} \frac{\varepsilon}{k} \overline{\rho u_i} \tag{46}$$

With $C_1^{'}$=3.2 is constant of Rotta [33], which translates speed with which the return to the isotropy of an initially anisotropic turbulence is carried out.

$\phi^2_{i\rho}$ is the linear term with the gradients mean velocity which is written:

$$\phi^2_{i\rho} = 0.8\overline{\rho u_k} U_{i,k} - 0.2\overline{\rho u_k} U_{k,i} \tag{47}$$

4. Discussion of results

4.1 Non-dimensional equations

The previous differential equations are castled in non dimensional forms when non dimensional parameters b_{11}, b_{22}, b_{12}, b_{13}, b_{23}, η and ε/KS are introduced for the kinematic field. In this section, all non dimensional parameters substitute previous turbulent ones $\overline{u_i u_j}$, $\overline{u_i \rho}$, K, ε and $\overline{\rho^2}$. A closed system of non-dimensional parameters can be obtained by casting equations (23)–(33) in non-dimensional forms and introducing the non-dimensional time *τ* = *St*. So transport equations for the above-mentioned parameters can be obtained:

$$\begin{aligned}\frac{db_{11}}{d\tau} &= -2\sin\theta\, b_{12} - 2\cos\theta\, b_{13} + \frac{\phi_{11}}{2kS} - \frac{\varepsilon}{3kS} \\ &+ \left(b_{11} + \frac{1}{3}\right)\left(2\sin\theta\, b_{12} + 2\cos\theta\, b_{13} + F_3 + \frac{\varepsilon}{kS}\right)\end{aligned} \tag{48}$$

$$\frac{db_{22}}{d\tau} = \frac{\phi_{22}}{2kS} - \frac{\varepsilon}{3kS} + \left(b_{22} + \frac{1}{3}\right)\left(2\sin\theta\, b_{12} + 2\cos\theta\, b_{13} + F_3 + \frac{\varepsilon}{kS}\right) \tag{49}$$

$$\begin{aligned}\frac{db_{12}}{d\tau} &= -\sin\theta\left(b_{22} + \frac{1}{3}\right) - \cos\theta\, b_{23} + \frac{\phi_{12}}{2kS} + 2\sin\theta\, b_{12}^2 \\ &+ 2\cos\theta\, b_{12} b_{13} + F_3 b_{12} + \left(\frac{\varepsilon}{kS}\right) b_{12}\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{db_{13}}{d\tau} &= -\sin\theta\, b_{23} - \cos\theta\left(b_{33} + \frac{1}{3}\right) - \frac{1}{2}F_1 + \frac{\phi_{13}}{2kS} + 2\cos\theta\, b_{13}^2 \\ &+ 2\sin\theta\, b_{12} b_{13} + \left(F_3 + \frac{\varepsilon}{kS}\right) b_{13}\end{aligned} \tag{51}$$

$$\frac{db_{23}}{d\tau} = -\frac{1}{2}F_2 + \frac{\phi_{23}}{2kS} + 2\cos\theta\, b_{23} b_{13} + 2\sin\theta\, b_{12} b_{23} + \left(F_3 + \frac{\varepsilon}{kS}\right) b_{23} \tag{52}$$

$$\frac{d\eta}{d\tau} = -F_3 + \eta\left(2\sin\theta\, b_{12} + 2\cos\theta\, b_{13} + F_3 - \frac{\varepsilon}{kS}\right) \quad (53)$$

$$\frac{d}{d\tau}\left(\frac{\varepsilon}{kS}\right) = -2C_{\varepsilon 1}\left(\frac{\varepsilon}{kS}\right)b_{13} + (1 - C_{\varepsilon 2})\left(\frac{\varepsilon}{kS}\right)^2 - C_{\varepsilon 1}(1 - C_{\varepsilon 3})\left(\frac{\varepsilon}{kS}\right)F_3 + \left(\frac{\varepsilon}{kS}\right)(2\sin\theta\, b_{12} + 2\cos\theta\, b_{13} + F_3) \quad (54)$$

In this level of work, the numerical resolution of the three systems of the non-linear differential equations , relating to the second-order models, parameterized by the Richardson number Ri and parameterized by the adimensional number ε/KS. The initial conditions of the results of the direct numerical simulation of Jacobitz et al. [19] have been retained and the fourth order Runge Kutta method has been used for integrating systems. The numerical integration has been conducted separately for the values 0.04, 0.08, 0.12, 0.16 and 0.20 of the non dimensional Richardson number Ri. The principal results are presented in the following paragraph.

4.2 Numerical integration and results

We start with the case of purely inclined shear. This case corresponds to θ=π/4 in the non-dimensional equation (23)-(33). A fourth order Runge-Kutta method is used for integrating the non-dimensional system of 10 non-linear differential equations.

The main objective of the present study is to determine the asymptotic behavior for a turbulence stratified presenting a inclined shear (θ=π/4), we must show the existence of the asymptotic equilibrium states of the models (SSG-LRR), (SSG-SL) and (SSG-CL) and we make a comparison between results obtained and result of the direct numerical simulation (DNS) of Jacobitz et al. [19].

The principal obtained results are presented in terms of the principal component of anisotropy b_{12}, the evolution of the ratios K/E and K_ρ/E, where E is total energy ($E=K+K_\rho$). A general tendency to equilibrium states for the dimensionless parameters has been observed at long time evolution and confirm qualitatively one of the principal results obtained by the DNS of Jacobitz et al. [19].

Table1: Equilibrium states for the principal component of anisotropy $(b_{12})_\infty$ predicted by different models and with ε/KS=0.5

b_{12}	DNS	SSG-LRR	SSG-CL	SSG-SL
Ri=0.0	...	-0.142	-0.140	-0.0639
Ri=0.05	...	-0.130	-0.136	-0.0581
Ri=0.15	...	-0.107	-0.129	-0.0570
Ri=0.2	-0,128	-0,097	-0,126	-0.0633
Ri=0.4	...	-0.0741	-0.110	-0.0555
Ri=0.6	...	-0.0678	-0.0950	-0.0553
Ri=1.0	...	-0.0612	-0.0731	-0.0556

However and as shown in Table 1, the coupling between the SSG model for the kinematic field and the CL model for the scalar field has shown the best agreement with the retained results of DNS of Jacobitz for the principal component of anisotropy (b_{12}). This agreement is observed for the non dimensional Richardson number Ri=0.2. The coupling between the SSG and the LRR model has also shown a tendency to an equilibrium states for this parameter but the values predicted are approximately two times larger than the values of the DNS of Jacobitz [23]. The Coupling between the SSG model and the SL model has predicted at high stratification very large equilibrium values for this non dimensional parameter. Obtained results are better than those obtained in the previous study of Bouzaiane et al. [34] when the LRR, SL and the CL second order models respectively are retained for both kinematic and scalar fields. Therefore the models of the second degree of SSG improved of the forecasts of three models.

In figure 1 is presented the time of evolution according to the time of the component b_{12} of the principal component of anisotropy b_{12}. Three models SSG-SL, SSG-CL and SSG-LRR confirm here also the existence of an asymptotic equilibrium states for the compoen b_{12}. However, we note that SSG-CL model shows a better agreement with values of DNS of Jacobitz [1] for non-dimensional time less than 30 ($\tau \geq 30$). Whereas the values predicted by two other models (SSG-SL and SSG-LRR) are lower approximately 40% than those of the DNS of Jacobitz. L'ordre de grandeur de valeurs prévues par des modèles est dans la même gamme des valeurs du DNS de Jacobitz [1].

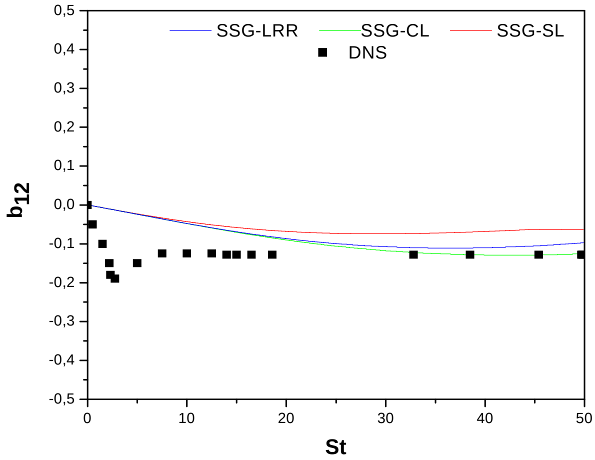


Figure 1: Time evolution of the component b_{12} for Ri=0.2 and ε/KS=0.5

Let us now examine predictions of models for the principal component of anisotropy b_{12}. Predictions of models at long time evolution are presented at figure 2. Here also the three models predict asymptotic equilibrium states at long time evolution and for the different retained values 0.00, 0.05, 0.15, 0.2, 0.4, 0.6 and 1.0 of the non dimensional Richardson number Ri.

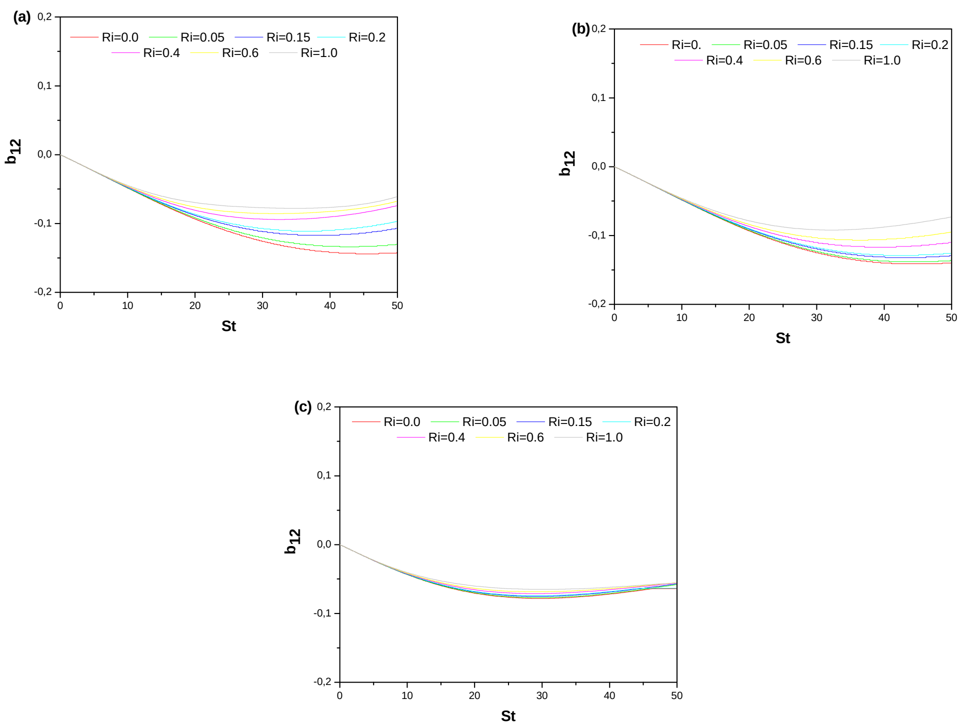


Figure 2: Time evolution of the component b_{12} according to SSG-LRR (a), SSG-CL (b) and SSG-SL (c)

Figures 2 (a), (b) and (c) show the evolution of the principal component of anisotropy b_{12} as a function of the normalized time St=τ, obtained by the SSG-LRR, SSG-CL and SSG-SL models respectively. Three models confirm the existence of an asymptotic equilibrium states for the component b_{12}. Three models indicate also that $(b_{12})_{oo}$ grows with Ri growing from 0.00 to 1.0. We note also that the asymptotic state for the SSG-LRR model and the SSG-CL model are reached very quickly compared to the prediction of the SSG-SL model.
A surprising result is observed, the coupling SSG-LRR and SSG- CL reproduce qualitatively growth of (b_{12}) with growing Ri from 0.00 to 1.0.

Thereafter we study in table 2 the influence of the Richardson number Ri on the kinetic energy K and the potential energy K_ρ for Ri=1.0 and Ri=2.0 according for the three retained second-order models SSG-LRR, SSG-CL, SSG-SL respectively.

Table 2: Asymptotic equilibrium values of non-dimensional parameters K/E and K_ρ/E for the inclined shear (θ=π/4)

	$(K/E)_\infty$				$(K_\rho/E)_\infty$			
Ri	DNS	SSG-LRR	SSG-CL	SSG-SL	DNS	SSG-LRR	SSG-CL	SSG-SL
0.2	...	0.91	0.96	0.94	...	0.087	0.03	0.05
0.4	...	0.79	0.91	0.89	...	0.2	0.08	0.1
0.6	...	0.69	0.85	0.81	...	0.3	0.14	0.18
1.0	0.6	0.56	0.75	0.77	0.4	0.43	0.24	0.22
2.0	0.52	0.48	0.64	0.63	0.42	0.51	0.35	0.36

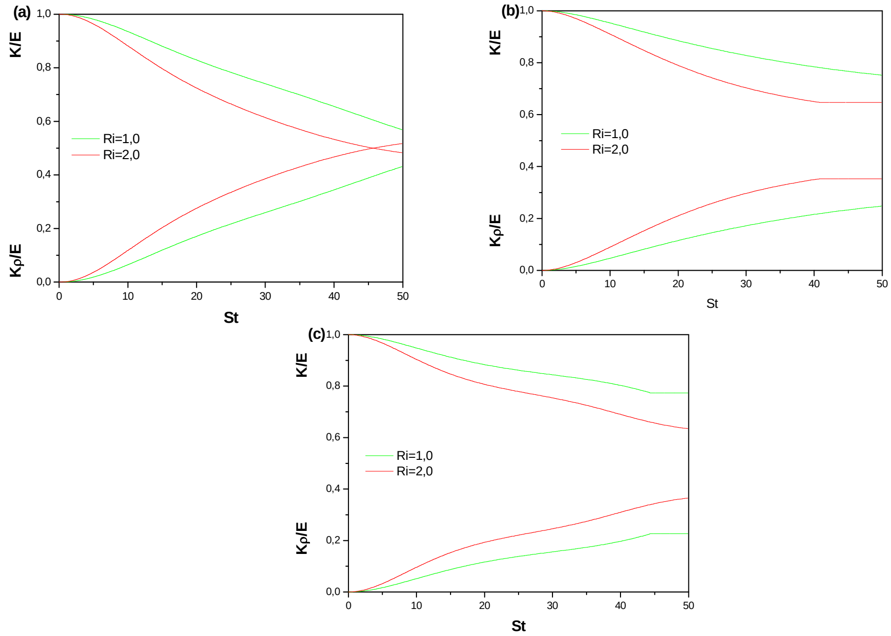
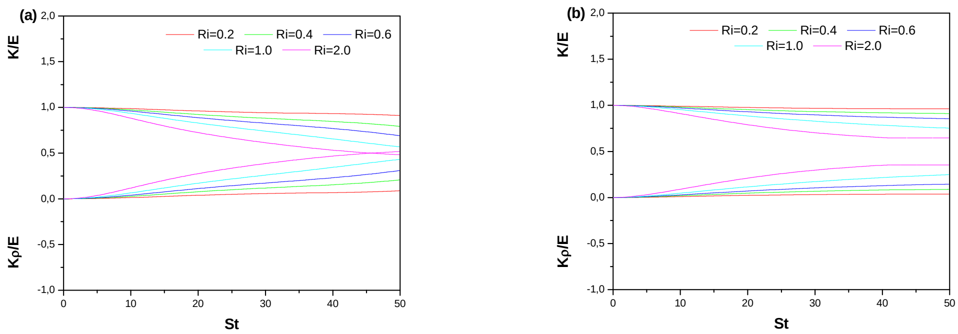


Figure 3: Time evolution of the ratios K/E and K_ρ/E according to SSG-LRR (a), SSG-CL (b) and SSG-SL (c)

In figure 3, we present according to time St=τ, the evolution of the ratios K/E and K_ρ/E, where E is total energy ($E=K+K_\rho$). The asymptotic equilibrium values of K/E and K_ρ/E are slightly different from the mean value 0.5. The second order models seem to divide total energy E into equal parts between the kinetic and potential terms. Where these ratios are obtained to leave adimensional ratio η which are written in the following forms:

$$\frac{K}{E}=\frac{1}{1+\eta} \quad \text{and} \quad \frac{K_\rho}{E}+\frac{\eta}{1+\eta} \tag{55}$$

Evolutions of the normalized turbulent energy K/E and K_ρ/E according to both non dimensional time St=τ and the non dimensional Richardson number Ri are presented in Figures 4 (a), (b) and (c) are relative to SSG-LRR, SSG-CL and SSG-SL respectively.



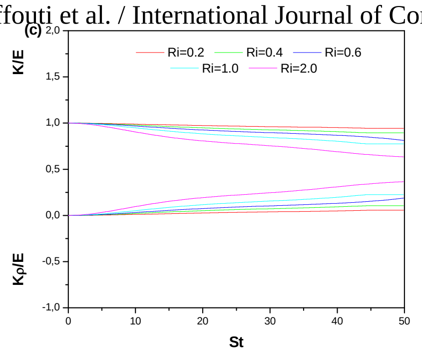


Figure 4: Time evolution of the ratios K/E and K_ρ/E according to SSG-LRR (a), SSG-CL (b) and SSG-SL (c) for differentes values of Richardson

In figures 4 (a), (b) and (c) are shown the time evolution of the ratio K/E and K_ρ/E for the mentioned values of Ri. Three models confirm here also the existence of an asymptotic equilibrium states for the non-dimensinal ratios for the ratios K/E and K_ρ/E. For K_ρ/E, the asymptotic states is reached here more quickly at high stratification (Ri=1.0 and 2.0) than at low stratification (Ri=0.2, 0.4 and 6.0). Furthermore, it is clear from these figures that for the CL model, the asymptotic equilibrium states are slightly different for cases of high stratification.

Thereafter, we propose to study the influence of the Richardson number Ri on the evolution of rate of non-dimensional shear ε/KS. In figures 5 (a), (b) and (c), we observe the evolution of ε/KS according to non-dimensional time τ=St relating to the SSG-LRR, SSG-CL and SSG-SL models for the values of Ri equal respectively to 0.0, 0.05, 0.15, 0.2, 0.4, 0.6 and 1.0. The SSG-LRR model confirm the existence of an asymptotic equilibrium states for the component ε/KS and that the values of equilibrium $(\varepsilon/KS)_{oo}$ decrease when Ri believes of the value 0.0 (which corresponds to a weak stratification) in the value 1.0 (which corresponds to a strong stratification). The asymptotic states of equilibrium is reached more quickly for SSG-SL more than that of other models SSG-CL.

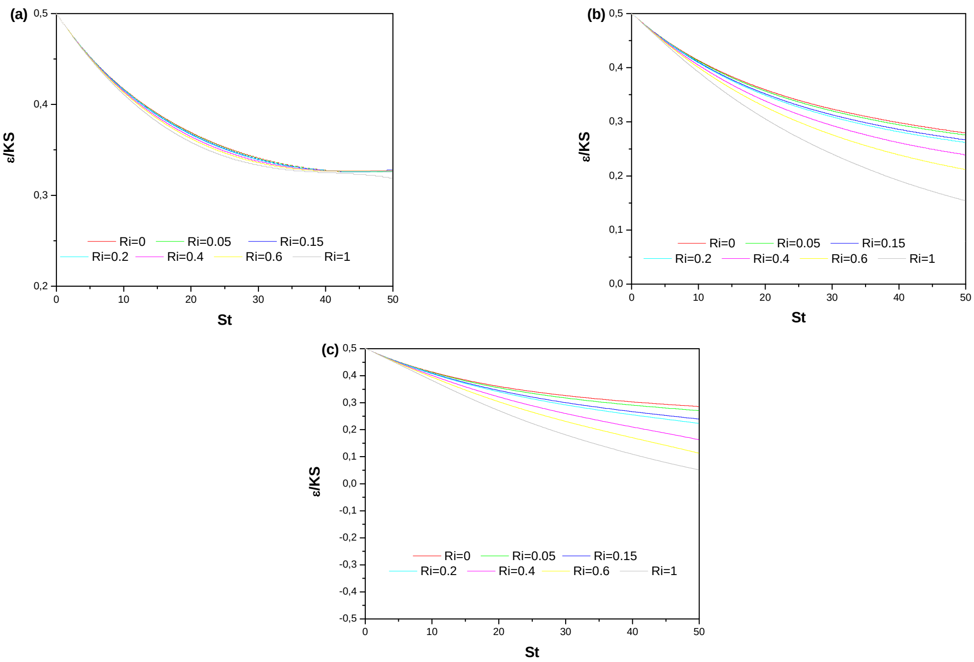
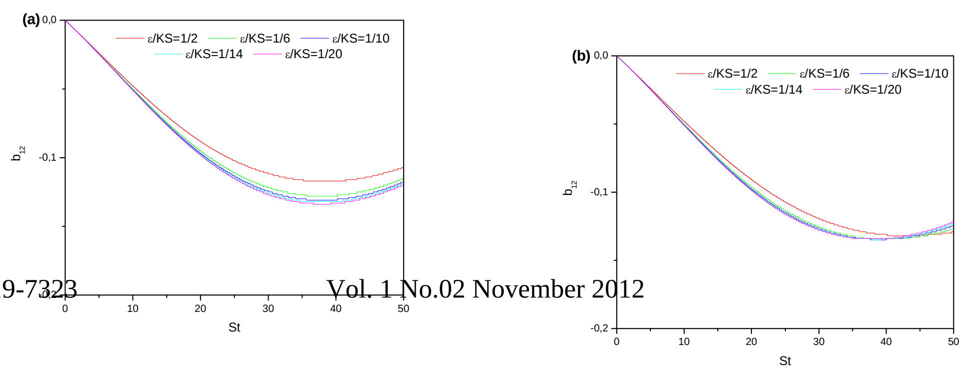


Figure 5: Evolution of ε/KS for different values of Ri according to SSG-LRR (a),SSG-CL (b) and SSG-SL (c)

The influence of the dimensionless shear number ε/KS on the principal component of anisotropy b_{12} is observed in figures 6 (a), (b) and (c).



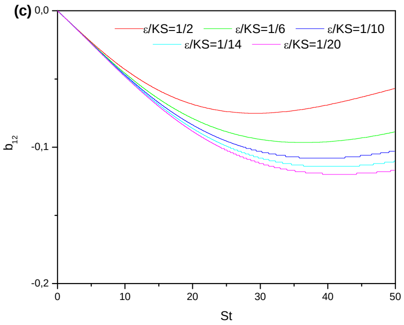


Figure 6: Evolution of the component b_{12} acording to SSG-LRR model (a), SSG-CL model (b) and SSG-SL model (c)

Figures 6 (a), (b) and (c) show the evolution of the principal component of anisotropy b_{12} as a function of the normalized time St=τ, obtained by the SSG-LRR, SSG-CL and SSG-SL for differnts values of ε/KS which varied of 1/20 until the 1/2. Three models confirm the existence of an asymptotic equilibrium states for the component b_{12}. We note that when the values of ε/KS decrease by 1/2 until the 1/20, the tensor of anisotropy of Reynolds decreases too for SSG-SL, SSG-CL and SSG-LRR models.

Conclusion

In the present study a stably stratified sheared turbulence has been investigated using second order modelling of equation of second order moments. The second order model of SSG has been retained for the kinematic field whereas three well known models are retained for the scalar field respectively. Modelled equations are castled in a non dimensional form when non dimensional kinematics and scalar parameters are introduced. Three non linear differential systems submitted to initial conditions of the results of DNS of Jacobitz have been integrated using the fourth order Rung-kutta method advanced to long time evolution. A general tendency to asymptotic equilibrium states has been observed. The SSG model is shown to be of a positive contribution in predicting asymptotic equilibrium values compared to our previous results [34]. Improve in predicting asymptotic equilibrium states has been essentially observed for the principal component of anisotropy b_{12}, for the normalized turbulent energy K/E and K_ρ/E and for the component ε/KS. We think that a correction of model coefficients by introducing the non dimensional Richardson number could be an important way to improve the model predictions for the component b_{12} of the anisotropic tensor. The effect of the non dimensional shear number [35] ε/*KS* on the evolution of a stably stratified turbulence and the Study of a stably stratified turbulence submitted to an inclined shear by second order models seem to be important directions of investigations. We point out here that in contrast with the results of direct numerical simulations where Jacobitz et al. [1] have studied the effect of the turbulent Reynolds number on the evolutions of turbulent parameters, the SSG second order model and other retained models do not take into account of the importance of such parameter. This can be considered as a border of the second order modelling retained in the present work.

References

[1] H. Ben Abdallah, M. Bouzaiane, T. Lili, A study of equilibrium states of homogeneous turbulence with a stable stratification and a non vertical shear, Phys.Chem.News, 23, pp. 34-46, (2005).
[2] B. E. Launder, G. Reece, W. Rodi, Progress in the development of a Reynolds stress closure, J. Fluid Mech. 68, pp. 537–576, (1975).
[3] B. E. Launder, S. Fu. Tselepidakis, Accomodating the effects of high strain rates in the modeling the pressure strain correlation, The University of Manchester of Science and Technology, March T.F. D/87/5/, (1987).
[4] T. J. Craft, B. E. Launder, A model for the pressure scalar gradient correlation and its application to homogeneous and inhomogeneous shear flow, Turbulent Shear Flow 7, Stanford University, USA, 12, pp. 1-6, (1989).
[5] T. H. Shih, J. L. Lumley, Second order modeling of scalar in turbulent shear flow, in Center of Turbulent Research 27th Aerospace Meeting, January 9–12, Nevada, (1989).
[6] Speziale C G, Sarkar S and Gatski T, Modeling the rapid pressure–strain correlation of turbulence, ICASE Report no 90-5, Contract Nas-18605, (1990).
[7] C. G. Speziale, N. M. Mhuiris, On the prediction of equilibrium states in homogeneous turbulene, J. Fluid Mech. 209, pp. 591–615, (1989).
[8] T. Von Karman, The Fundamentals of the Statistical Theory of Turbulence, J. Aero Sci., Vol. 4, pp. 131-138, , (1937).
[9] W . G. Rose, Results of an Attempt to Generate a Homogeneous Turbulent Shear Flow, J. Fluid Mech., Vol. 25, pp. 97-120, (1966).
[10] F. H. Champagne, V. G. Harris, and S. Corrsin, Experiments on Nearly Homogeneous Turbulent Shear Flow, J. Fluid Mech., Vol. 41, pp. 81- 139, (1970).
[11] V. G. Harris, J. A. Graham, and S. Corrsin, Further Experiments in Nearly Homogeneous Turbulent Shear Flow, J. Fluid Mech., Vol. 81, pp. 657-687, (1977).
[12] C. G. Speziale, S. Sarkar, T. B. Gatski, Modeling the pressure strain correlation of turbulence: An invariant dynamical system approach. J. Fluid Mech. 227, 245, (1991).
[13] C. G. Speziale, T. B. Gatski, Analysis and modeling of anisotropies in the dissipation rate of turbulence. J. Fluid Mech. 344, 155, (1997).

[14] T. Gerz, U. Shumann, S. Elghobachi, Direct numerical simulation of stratified homogeneous turbulent shear flow, J.Fluid Mech.200, pp. 563-594, (1989).

[15] S. E. Holt, J. R. Koseff, J. H. Ferziger, A numerical study of the evolution and structure of homogeneous turbulent shear flows. J. Fluid Mech. 237, 499, (1992).

[16] C. G. Speziale, S. Sarkar, T. B. Gatski, Modelling the pressure–strain correlation of turbulence: an invariant dynamical systems approach. J. Fluid Mech. 227, pp. 245–272, (1991).

[17] R. Abid, C. G. Speziale, Predicting equilibrium states with Reynolds stress closures in channel flow and homogeneous shear flow. Phys. Fluids A 5, pp. 1776–1782, (1993).

[18] S. Tavoularis, S. Corrsin, Experiments in nearly homogeneous turbulent shear flows with a uniform mean temperature gradient. Part 1. J. Fluid Mech. 104, pp. 311–347, (1981).

[19] S. Holt, J. Ferziger, A numerical study of the evolution and structure of homogeneous stably stratified sheared turbulence, J.Fluid Mech.237, pp. 499-539, (1992).

[20] F. Jacobitz, S. Sarkar, The effect of non vertical shear on turbulence in a stably stratified medium, Phys.Fluid.10, pp. 1158-1168, (1998).

[21] S. Komori, H. Ueda, F. Ogino and T. Mizushina, Turbulence structure in stably stratified open-channel flow, J.Fluid Mech.130, 13, (1983).

[22] J. J. Rohr, E. C. Itsweire, K. N. Helland, C. W. Van Atta, Growth and decay of turbulence in a stably stratified shear flow, J.Fluid Mech.195, pp. 65-77, (1988).

[23] H. Ben Abdallah, M. Bouzaiane, T. Lili, A study of equilibrium states of homogeneous turbulence with a stable stratification and a non vertical shear, Phys.Chem.News, 23, pp. 34-46, (2005).

[24] F. Jacobitz, A comparaison of the turbulence evolution in a stratified fluid with vertical or horizontal shear, Journal of Turbulence 3, (2002).

[25] M. Bouzaiane, H. Ben Abdallah and T. Lili, A second order modeling of a stably stratified sheared turbulence submitted to a non vertical shear. Journal of turbulence (2004).

[26] C. G. Speziale, S. Sarkar, T. B. Gatski, Modeling the rapid pressure-strain correlation of turbulence. Phys. Fluids 84, pp. 85-95, (1990).

[27] U. Schuman, Realisability of Reynolds stress turbulence models, Phys. of Fluids 20, pp. 721-728, (1977).

[28] T. H. Shih, J. L. Lumley , Realisability in second moment closures revisited NASA TM. 10646 (1994).

[29] O. Zeman, J. L. Lumley, Modeling buoyancy driven mixed layers, J. Atm Sci. 33, pp. 1974-1988, (1976).

[30] T. H. Shih, J. L. Lumley , Realisability in second moment closures revisited NASA TM. 10646 (1994).

[31] B. E. Launder, Principles and performance of TCL Based second Moment closures, Commemorative conference HA MINH THIEU, pp. 1-19, Toulouse November (1999).

[32] B. E. Launder, G. Reece et W. Rodi**,** Advanced turbulence models for industrial application, in turbulence transition modeling edited by M.Hallback and D.S.Hennigson and A.V.Johansson and P.H.Alfredsson, Dordrecht, (1996).

[33] A. Cadiou, Contribution à l'étude de modèle de turbulence au second ordre, thèse de doctorat, école centrale de Nantes, (1996).

[34] M. Bouzaiane., H. B. Abdallah T. Lili, A study of the asymptotic behaviour in stably stratified sheared turbulence, Journal of Turbulence 4, (2003).

[35] F. G. Jacobitz, On the shear number effect in stratified shear flow. Theoretical and Computational Fluid Dynamics 13, pp. 171-188, (1999).