Ranking DMUs with Fuzzy Data Using MAJ model

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Abstract_ Data Envelopment Analysis (DEA) is a non-parametrical method for evaluating the efficiency of Decision Making Units (DMUs) using mathematical programming. There are several methods for evaluating the rank of efficient DMUs such as Anderson and Peterson method (AP) or Mehrabian, Alirezaei and Jahanshahloomethod (MAJ).These models work with deterministic data. In case that data is fuzzy we should have an appropriate technique to solve the model.In this paper, a new method is proposed to find the rank of Decision Making Units (DMUs) when all of data were fully fuzzy via MAJ model. To illustrate the proposed model numerical example is solved.

Key Words-Data Envelopment Analysis (DEA), Ranking, Crisp Linear Programming (CLP), Fully Fuzzy Linear Programming (FFLP), Triangular Fuzzy Number.

I. INTRODUCTION

DEA is a powerful tool in estimating efficiency of decision making units with multiple inputs and outputs. Charnes, Cooper and Rhodes [3] were the pioneers of the field that introduced their first model named “CCR” in 1978. The assumption is that all the data have specific numerical values.

Anderson and Peterson (A.P) [1] proposed a method for ranking efficient units on basis of the position of each eliminated efficient DMUs in relation to its corresponding new Production Possibility Set (PPS). In 1999, Mehrabian and his coworkers [2] proposed a technique, MAJ to confront the AP techniques problem; these problems include infeasibility of covering form in case of data with special structure and instability that causes sudden mutation on performance of some under survey DMUs when removing some of DMUs. These problems include infeasibility when the data’s have special structure and non-stability in case the elimination of some DMUs causes sudden mutation on efficiency of an evaluated DMU. There are several method for ranking efficient units with stochastic data, including Nabahat et al. [8], also ranking using AP technique of Razavyan, and Tohidi [4]. Hosseinzadeh Lotfi et al. [5] ranking using coefficient variation. In 2010 Kumar et al. [9] find the fuzzy optimal solution of fully fuzzy linear programming (FFLP) problems with inequality constraints. By using his method the fuzzy optimal solution of FFLP problems with inequality constraints, occurring in real life situation, can be easily obtained. Nabahat et al. [10] rank DMUs using Ideal and Anti-ideal Decision-Making Units with Fuzzy Data. In this paper we want to rank fully fuzzy DMUs by using MAJ model. It is clear that MAJ model is a type of linear programming and fuzzy MAJ model is a fully fuzzy linear programming problem. Thus, by converting it into crisp linear programming we can defuzzify the model.

II. PRELIMINARIES

A. Basic Definitions

In this section some necessary backgrounds and notions of fuzzy set theory are reviewed.

Definition 2.1 [6] A triangular fuzzy number (a, b, c) is said to be non-negative fuzzy number if $a \geq 0$.

Definition 2.2 [7] A ranking function is a function $\Re: F(R) \rightarrow R$, where $F(R)$ is a set of fuzzy numbers defined on set of real numbers, which maps each fuzzy number into the real line, where a natural order exists. Let $\widetilde{A} = (a, b, c)$ be a triangular fuzzy number then $\Re(\widetilde{A}) = \frac{a+2b+c}{4}$

Definition 2.3 Let $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers, then

(i) $\widetilde{A} \precsim \widetilde{B}$ iff $a_1 \leq a_2,\ b_1 - a_1 \leq b_2 - a_2,\ c_1 - b_1 \leq c_2 - b_2$.

(ii) $\widetilde{A} \succsim \widetilde{B}$ iff $a_1 \geq a_2,\ b_1 - a_1 \geq b_2 - a_2,\ c_1 - b_1 \geq c_2 - b_2$.

(iii) $\widetilde{A} = \widetilde{B}$ iff $a_1 = a_2, b_1 = b_2, c_1 = c_2$.

B. Arithmetic Operations

In this section, arithmetic operations between two triangular fuzzy numbers, defined on universal set of real numbers R, are reviewed [6].

Let $\widetilde{A} = (a, b, c)$ and $\widetilde{B} = (e, f, g)$ be two triangular fuzzy numbers then

(i) $\widetilde{A} \oplus \widetilde{B} = (a, b, c) \oplus (e, f, g) = (a+e, b+f, c+g)$

(ii) $-\widetilde{A} = -(a, b, c) = (-c, -b, -a)$

(iii) $\widetilde{A} \ominus \widetilde{B} = (a, b, c) - (e, f, g) = (a-g, b-f, c-e)$

(iv) Let $\widetilde{A} = (a, b, c)$ be any triangular fuzzy number and $\widetilde{B} = (x, y, z)$ be a non-negative triangular fuzzy number then

$$\widetilde{A} \otimes \widetilde{B} \simeq \begin{cases} (ax, by, cz), & a \geq 0 \\ (az, by, cz), & a < 0, c \geq 0 \\ (az, by, cx), & c < 0 \end{cases}$$

III. Fully Fuzzy MAJ Model

Assume that there are n decision-making units (DMUs) to be evaluated, each DMU with m inputs and s outputs. We denote the inputs and outputs of $(DMU_j)\ j = 1,2,\dots,n$ with $\tilde{x}_{ij}$, $i = 1,2,\dots,m$ and $\tilde{y}_{rj}$, $r = 1,2,\dots,s$ which all of the inputs and outputs are fuzzy symmetrical triangular numbers and positive.

MAJ model with crisp data is shown below:

$$\begin{aligned} & \min 1 + w_o - v_o \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_{ij} \leq x_{io} + w_o - v_o \quad i = 1,\dots,m,\ j = 1,\dots,n \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_{rj} \geq y_{ro} \quad r = 1,\dots,s \qquad (1) \\ & \sum_{j=1}^{j=1n} \lambda_j = 1 \\ & \lambda_j \geq 0, w_o, v_o \geq 0 \end{aligned}$$

where x_{ij}, $i = 1,2,\dots,m$ and y_{rj}, $r = 1,2,\dots,s$ in (1) respectively denotes the inputs and outputs of $(DMU_j)\ j = 1,2,\dots,n$.

The MAJ model with fully fuzzy data can formulated as:

$$\begin{aligned} & Min \quad 1 \oplus \widetilde{w_o} \ominus \widetilde{v_o} \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \widetilde{\lambda_j} \otimes \widetilde{x_{ij}} \precsim \widetilde{x_{io}} \oplus \widetilde{w_o} \ominus \widetilde{v_o} \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \widetilde{\lambda_j} \otimes \widetilde{y_{rj}} \succsim \widetilde{y_{ro}} \qquad (2) \\ & \sum_{j=1}^{n} \widetilde{\lambda_j} = 1 \\ & \widetilde{\lambda_j}, \widetilde{w}_o, \widetilde{v}_o \geq 0 \end{aligned}$$

IV. Method to Convertthe Inequality Constraintsinto Equality Constraints

The following method is used to convert the inequality constraints into equality constraints.

In case $\sum_{\substack{j=1 \\ j\neq o}}^{n} \widetilde{\lambda_j} \otimes \widetilde{x_{ij}} \precsim \widetilde{x_{io}} \oplus \widetilde{w_o} \ominus \widetilde{v_o}$ Convert such type of inequality constraints into equality constraints by introducing non-negative variables $\widetilde{S_i}$ to the left side of the constraints i.e.

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \widetilde{\lambda_j} \otimes \widetilde{x_{ij}} \oplus \widetilde{S_i} = \widetilde{x_{io}} \oplus \widetilde{w_o} \ominus \widetilde{v_o}$$

where $\widetilde{S_i}$ is a non-negative fuzzy number.

In case $\sum_{\substack{j=1 \\ j\neq o}}^{n} \widetilde{\lambda_j} \otimes \widetilde{y_{rj}} \gtrsim \widetilde{y_{ro}}$ Convert such type of inequality constraints into equality constraints by introducing non-negative variables $\widetilde{S_r}$ to the right side of the constraints i.e.

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \widetilde{\lambda_j} \otimes \widetilde{y_{rj}} = \widetilde{y_{ro}} \oplus \widetilde{S_r}$$

where $\widetilde{S_r}$ is a non-negative fuzzy number.

If all the parameters $\widetilde{\lambda_j}, \widetilde{x_{ij}}, \widetilde{y_{rj}}, \widetilde{w_o}, \widetilde{v_o}, \widetilde{S_i}, \widetilde{S_r}$ are represented by triangular fuzzy numbers $(\alpha_j, \beta_j, \gamma_j), (a_{ij}, b_{ij}, c_{ij}), (d_{rj}, e_{rj}, f_{rj}), (w_a, w_b, w_c), (v_a, v_b, v_c)$, $(s_i, t_i, u_i), (\acute{s}_r, \acute{t}_r, \acute{u}_r)$ Respectively then the FFLP problem, obtained in step "above", may be written as:

$$Min \quad (1,1,1) \oplus (w_a, w_b, w_c) \ominus (v_a, v_b, v_c)$$

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} (\alpha_j, \beta_j, \gamma_j) \otimes (a_{ij}, b_{ij}, c_{ij}) \oplus (s_i, t_i, u_i) = (a_{io}, b_{io}, c_{io}) \oplus (w_a, w_b, w_c) \ominus (v_a, v_b, v_c)$$

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} (\alpha_j, \beta_j, \gamma_j) \otimes (d_{rj}, e_{rj}, f_{rj}) = (d_{ro}, e_{ro}, f_{ro}) \oplus (\acute{s}_r, \acute{t}_r, \acute{u}_r) \qquad (3)$$

$$\sum_{j=1}^{n} (\alpha_j, \beta_j, \gamma_j) = (1,1,1)$$

$$w_a \geq 0, w_b - w_a \geq 0, w_c - w_b \geq 0$$
$$v_a \geq 0, v_b - v_a \geq 0, v_c - v_b \geq 0$$
$$\alpha_j \geq 0, \beta_j - \alpha_j \geq 0, \gamma_j - \beta_j \geq 0$$

Using arithmetic operations, definition 2.3, the FFLP, obtained in (3) is converted into the following problem:

$$Min \quad \Re(1 + w_a - v_c, 1 + w_b - v_b, 1 + w_c - v_a)$$

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} (\alpha_j a_{ij} + s_i, \beta_j b_{ij} + t_i, \gamma_j c_{ij} + u_i) = (a_{io} + w_a - v_c, b_{io} + w_b - v_b, c_{io} + w_c - v_a)$$

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} (\alpha_j d_{rj}, \beta_j e_{rj}, \gamma_j f_{rj}) = (d_{ro} + \acute{s}_r, e_{ro} + \acute{t}_r, f_{ro} + \acute{u}_r) \qquad (4)$$

$$\sum_{j=1}^{n} (\alpha_j, \beta_j, \gamma_j) = (1,1,1)$$

$$w_a \geq 0, w_b - w_a \geq 0, w_c - w_b \geq 0$$
$$v_a \geq 0, v_b - v_a \geq 0, v_c - v_b \geq 0$$
$$\alpha_j \geq 0, \beta_j - \alpha_j \geq 0, \gamma_j - \beta_j \geq 0$$
$$s_i \geq 0, t_i - s_i \geq 0, u_i - t_i \geq 0$$
$$\acute{s}_r \geq 0, \acute{t}_r - \acute{s}_r \geq 0, \acute{u}_r - \acute{t}_r \geq 0$$

Using definition 2.2 of the proposed method the above FFLP problem is converted into the following (crisp linear programming) CLP problem:

$$Min \quad \frac{1 + w_a - v_c + 2 + 2w_b - 2v_b + 1 + w_c - v_a}{4}$$

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \alpha_j a_{ij} + s_i = a_{io} + w_a - v_c$$
$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \beta_j b_{ij} + t_i = b_{io} + w_b - v_b$$
$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \gamma_j c_{ij} + u_i = c_{io} + w_c - v_a \qquad (5)$$
$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \alpha_j d_{rj} = d_{ro} + \acute{s}_r$$
$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \beta_j e_{rj} = e_{ro} + \acute{t}_r$$
$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \gamma_j f_{rj} = f_{ro} + \acute{u}_r$$
$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \alpha_j = 1$$
$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \beta_j = 1$$
$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \gamma_j = 1$$
$$w_a \geq 0\,, w_b - w_a \geq 0\,, w_c - w_b \geq 0$$
$$v_a \geq 0\,, v_b - v_a \geq 0\,, v_c - v_b \geq 0$$
$$\alpha_j \ \geq 0\,, \beta_j - \alpha_j \geq 0\,, \gamma_j - \beta_j \geq 0 \quad j = 1, \dots, n$$
$$s_i \geq 0 \ \,, t_i - s_i \geq 0\,, u_i - t_i \geq 0 \quad i = 1, \dots, m$$
$$\acute{s}_r \geq 0\,, \acute{t}_r - \acute{s}_r \geq 0\,, \acute{u}_r - \acute{t}_r \geq 0 \quad r = 1, \dots, s$$

V. Numerical Example

Suppose 5 decision-making units with two fuzzy inputs and outputs as shown in table 1, all of which have symmetrical triangular membership functions.

DMU (i)	1	2	3	4	5
Input 1	(5.9,6.5,7.1)	(4.4,4.9,5.4)	(2.9,2.9,2.9)	(3.4,4.1,4.8)	(3.5,4,4.5)
Input 2	(3.6,4.1,4.6)	(2.2,2.6,3)	(1.4,1.5,1.6)	(2.2,2.3,2.4)	(1.9,2.1,2.3)
Output 1	(4.4,5.1,5.8)	(2.7,3.2,3.7)	(2.2,2.2,2.2)	(2.5,2.9,3.3)	(2.4,2.6,2.8)
Output 2	(6.5,7.4,8.3)	(4.3,5.1,5.9)	(3.3,3.5,3.7)	(5.5,5.7,5.9)	(3.8,4.1,4.4)

Table 1.Decision making Units with fuzzy inputs and outputs

By applying model (5) and using Lingo software the rank of all DMUs can be evaluated that shown in table 2.

DMU (i)	1	2	3	4	5
Efficiency	9.75	2.12857	2.1	2.8375	0.86789
Rank	1	3	4	2	5

Table 2. Rank of DMUs with Fuzzy Data

Nabahat [10] and his coworker defuzzify the DMUs with fuzzy data using different$\alpha - levels$ and finally rank the efficient DMUs by using Ideal and Anti Ideal DMUs, through his method the rank of DMUsat different $\alpha - levels$ by using RC indexand " Best-Best", "Worst-Worst", "Best-Worst" and "Worst-Best" methods is as following tables.

Best-Best	α	DMU1	DMU 2	DMU 3	DMU 4	DMU 5
RANK	0	5	3	1	4	2
	0.25	5	2	1	4	3
	0.5	5	1	3	4	2
	0.75	5	1	3	2	4
	1	4	1	5	3	2

Table 3.Rank of DMUs using Best-Best model with Fuzzy Data

Worst-Worst	α	DMU1	DMU 2	DMU 3	DMU 4	DMU 5
RANK	0	3	1	5	2	4
	0.25	3	1	5	2	4
	0.5	3	1	5	2	4
	0.75	3	1	4	2	5
	1	4	1	5	3	2

Table 4. Rank of DMUs using Worst-Worst model with Fuzzy Data

Best-Worst	α	DMU1	DMU 2	DMU 3	DMU 4	DMU 5
RANK	0	4	2	5	3	1
	0.25	3	1	4	2	5
	0.5	5	1	4	3	2
	0.75	4	1	5	3	2
	1	4	1	5	3	2

Table 5. Rank of DMUs using Best-Worst model with Fuzzy Data

Worst- Best	α	DMU1	DMU 2	DMU 3	DMU 4	DMU 5
RANK	0	3	1	5	2	4
	0.25	3	1	5	2	4
	0.5	4	1	5	3	2
	0.75	4	1	5	2	3
	1	4	1	5	3	2

Table 6. Rank of DMUs using Worst-Best model with Fuzzy Data

VI. CONCLUSION

Data Envelopment Analysis (DEA) is a non-parametrical method for evaluating the efficiency of Decision Making Units (DMU) using mathematical programming. There are several methods foranalyzing the efficiency of Decision Making Units, among which are Charnes Cooper Rodes (CCR) andBanker Charnes Cooper (BCC), which compute the efficiency of Decision Making Units using the linear programming. All these calculations occur when all data, that is the inputs and the outputs of Decision Making Units, are positive and crisp data. There are several methods for ranking the efficient DMUs such as Anderson Peterson method (AP) or MAJ model. Table 2 shows the rank of DMUs with proposed method and tables 3 to 6show the rank of DMUs at different $\alpha - levels$ by "Best-Best", "Worst-Worst", "Best-Worst" and "Worst-Best" methods. The difference between the rank of DMUs and the various methods is clear.

Through applications, number of efficient units are more than one. Therefore managers want to find a suitable and analytical method in order to rank efficient units. In many situations we encounter DMUs with fuzzy data. Therefore in this paper we consider the DMUs with fully fuzzy data and rankingthe DMUsusing MAJ model that is FFLP problem and convert it into equivalent CLP problem.

REFERENCES

[1] P. Andersen, and N. C. Petersen, A procedure for ranking efficient units in data Envelopment analysis, Management Science 39, 1261-1264, 1993.

[2] S. Mehrabian, M. R. Alirezaei, and G. R. Jahanshahloo, A complete efficiency ranking of decision making units: an application to the teacher training university, Computational Optimization and Application 14, 1999.

[3] A. Charnes, W. W. Cooper, and E. Rhodes, Measuring the efficiency of decision making units, European Journal of Operational Research 2, 429-444, 1978.

[4] S.H. Razavyan, GH. Tohidi, Ranking of efficient DMUs with stochastic data, International Mathematical Forum, 3, 79-83, 2008.

[5] F. Hosseinzadeh Lotfi, N. Nematollahi, M.H. Behzadi, Ranking Decision making units with stochastic data by using coefficient of variation,Mathematical and Computational Application, 15, 148-155, 2010.

[6] A. Kaufmann and M. M. Gupta, Introduction to Fuzzy Arithmetic: Theory and Applications, Van Nostrand Reinhold, New York, 1985.

[7] T. S. Liou, M. J. Wang, Ranking Fuzzy Numbers with Integral Value, Fuzzy Sets and Systems, vol. 50, 1992, 247-255.

[8] M. Nabahat, F. Esmaeeli Sangari, S. M. Mansourfar, Ranking Decision Making Unit with Stochastic Data Using JAM Model, Journal of Mathematics Research, vol. 3,50-54, 2011.

[9] A. Kumar, J. Kaur, P. Singh, Fuzzy Optimal Solution of Fully Fuzzy Linear Programming Problems with Inequality Constraints, International Journal of Mathematical and Computer Sciences, 6:1, 37-41, 2010.

[10] M. Nabahat, F. Esmaeeli Sangari, Computing the Efficiency of Decision-Making Units with Fuzzy Data Using Ideal and Anti-Ideal Decision Making Units, International Journal of Computer Science and Information Security,Vol. 10, No.2, 2012.